

## CODES IN SPHERICAL CAPS

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**ABSTRACT.** We consider bounds on codes in spherical caps and related problems in geometry and coding theory. An extension of the Delsarte method is presented that relates upper bounds on the size of spherical codes to upper bounds on codes in caps. Several new upper bounds on codes in caps are derived. Applications of these bounds to estimates of the kissing numbers and one-sided kissing numbers are considered.

It is proved that the maximum size of codes in spherical caps for large dimensions is determined by the maximum size of spherical codes, so these problems are asymptotically equivalent.

### 1. INTRODUCTION

The subject of this paper is codes in spherical caps, i.e., packings of a metric ball on the surface of the sphere in  $\mathbb{R}^n$  (a spherical cap) with metric balls (caps) of a smaller radius. Codes in spherical caps are related to more familiar spherical codes and find a number of interesting applications in both classical and recent works.

Spherical cap codes have been used to derive an asymptotic upper bound on the maximum size of spherical codes and a bound on the packing density  $\Delta_n$  of the  $n$ -space by equal spheres, see Sidelnikov [24], Kabatiansky and Levenshtein [10] and Levenshtein [12]. More recently they have been used to derive upper bounds on the size of binary constant weight codes, see Agrell, Vardy and Zeger [1]. Even more recently, estimates of the maximum size of codes in a spherical cap have been used together with an extension of Delsarte's method to derive new estimates of the kissing number  $k(n)$  in small dimensions. In particular, a long-standing conjecture that  $k(4) = 24$  was solved in [16, 14] and a related problem for "one-sided kissing numbers" was solved in dimension 4 in [15].

In this paper we focus on the study of spherical cap codes rather than on their applications in related geometric problems of coding theory. More specifically, we study bounds on the size of spherical cap codes with a given angular separation and their relation to spherical codes.

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In Section 3 we recall a few known bounds on the size of codes in a spherical cap and spherical strip. In Section 4 we formulate an extension of Delsarte's bound on the size of spherical codes to cover the case of spherical caps. As usual, the polynomial involved in the computation of the bound must be expandable in a linear combination of Gegenbauer polynomials with nonnegative coefficients. A new condition in the theorem relates the values of the polynomial to constructions of codes in a spherical cap. The method described was used in [14] to prove that  $k(4) = 24$ . This link serves as an additional motivation for studying spherical cap codes.

In Section 5 we show that in the case of large code distance, the size of a code in a spherical cap can be exactly expressed via the size of codes on the entire sphere. The result is used to relate the size of spherical cap codes with the kissing number  $k(n)$ . We consider examples of small dimensions  $n = 3, 4$  and illustrate the application of the extended Delsarte's method to the derivation of the values of  $k(n)$  in these cases.

In Section 6 we derive a new bound on the size of spherical cap codes that relies on a transformation from codes in caps to codes on the hemisphere. In the same section we also address the problem of the maximum size of spherical cap codes in the case of large dimensions. A common perception in coding theory, originating with the asymptotic results of [10] is that codes in spherical caps are analogous to constant weight binary codes (i.e., codes formed of vectors with a fixed number of ones). Constant weight codes possess a rich combinatorial structure related to the properties of the Johnson graph [7]; however, no similar theory has arisen for the spherical case. We provide an explanation of this by showing that the asymptotic problem of constructing spherical cap codes is equivalent to the analogous problem for codes on the entire sphere.

Section 7 is devoted to a particular case of cap codes, namely codes in hemispheres. We derive an upper bound on the size of such codes and use it to derive estimates of a parameter closely related to  $k(n)$ , the so-called one-sided kissing number  $B(n)$ . We derive estimates of  $B(n)$  for  $n = 5, 6, 7, 8$  and make a conjecture about the exact values for some of these cases.

In Section 8 we use the method of Section 7 to derive another upper bound on spherical cap codes that often improves the result of Section 6. We show in examples that the bounds derived in this paper are sometimes better than the previously known results. In Section 9 we present a general approach to bounding the size of codes in spherical caps that combines several features of the methods introduced earlier in the paper. We conclude with a brief discussion of applications of the bounds on cap codes.

## 2. NOTATION AND PRELIMINARIES

Let  $S^{n-1}$  be a unit sphere in  $n$  dimensions and let  $e_n = (0, \dots, 0, 1)$  be the "North pole." Let  $0 \leq \psi \leq \phi \leq 90^\circ$ . Denote by

$$Z(n, [\psi, \phi]) = \{x \in S^{n-1} \mid \cos \phi \leq \langle x, e_n \rangle \leq \cos \psi\}$$

a strip cut on the sphere by two planes perpendicular to the vector  $e_n$ . In particular,  $\text{Cap}(n, \phi) = Z(n, [0, \phi])$  is a spherical cap with angular radius  $\phi$  drawn about  $e_n$ . A finite subset  $C \subset Z$  is called a code. Below by  $\text{dist}(\cdot, \cdot)$  we denote the angular distance between two points on the sphere. If a code  $C$  has minimum angular separation  $\theta$ , i.e., satisfies  $\text{dist}(x_1, x_2) \geq \theta$  for any two distinct points  $x_1, x_2 \in C$ ,

we call it a  $\theta$ -code. Let

$$A(n, \theta, [\psi, \phi]) = \max_{\substack{C \subset Z(n, [\psi, \phi]) \\ C \text{ a } \theta\text{-code}}} |C|$$

be the maximum size of a  $\theta$ -code in the strip  $Z$ . For spherical caps we will write  $A(n, \theta, \phi)$  instead of  $A(n, \theta, [0, \phi])$  and use a separate notation  $B(n, \theta) := A(n, \theta, \pi/2)$  for codes in the hemisphere  $S_+ := \text{Cap}(n, \pi/2)$ . In the case of  $\phi = \pi$  (the entire sphere) we will call  $C$  a spherical  $\theta$ -code and use the notation  $A(n, \theta)$  to denote its maximum possible size.

The quantity  $k(n) = A(n, \pi/3)$  is equal to the number of nonoverlapping unit spheres that can touch the sphere  $S^{n-1}$  and is called the *kissing number* in dimension  $n$ . The problem of finding or bounding  $k(n)$  has a rich history [4, 14, 19, 20].

Likewise, the quantity  $B(n) = B(n, \pi/3)$  is called the *one-sided kissing number*. The one-sided kissing number problem was considered recently in [2, 3].  $B(n)$  has the following geometric meaning. Let  $H$  be a closed half-space of the  $n$ -dimensional Euclidean space. Suppose that  $S$  is a unit sphere in  $H$  that touches the supporting hyperplane of  $H$ . The one-sided kissing number  $B(n)$  is the maximal number of unit nonoverlapping spheres in  $H$  that can touch  $S$ .

The function  $A(n, \theta)$  has received considerable attention in the literature. Therefore, one possible avenue of studying spherical cap codes is to map them on the sphere or hemisphere and relate them to spherical codes. In this paper, we rely on a number of mappings between spheres, spherical caps, and spherical strips to estimate the maximum size of a code in a spherical cap. Some of them have been used earlier in the literature while the others have not been emphasized in the context of estimating the code size. The main problem addressed here is to design the mappings so that the distance between the images of two points in the domain can be bounded in terms of the original distance. One often-used map is the orthogonal projection  $\Pi_n$  which sends the point  $x \in S^{n-1}$  along its meridian to the equator of the sphere, i.e., the set of points on  $S^{n-1}$  with  $x_n = 0$  ( $\Pi_n$  is defined on  $S^{n-1}$  without the North and South poles). Below we use the notation  $S^{n-2}$  to refer to the equator of the sphere  $S^{n-1}$ .

Throughout this paper we use the function  $\omega(\theta, \alpha, \beta)$  defined by

$$\cos \omega(\theta, \alpha, \beta) = \frac{\cos \theta - \cos \alpha \cos \beta}{\sin \alpha \sin \beta}.$$

In the case of  $\alpha = \beta$  we write  $\omega(\theta, \alpha)$  instead of  $\omega(\theta, \alpha, \alpha)$ . This function describes the change of the distance between two points on  $S^{n-1}$  which are  $\alpha$  and  $\beta$  away from  $e_n$  and  $\theta$  away from each other under the action of  $\Pi_n$ .

### 3. SPHERICAL STRIP (CAP) CODES AND SPHERICAL CODES

Several estimates on the size of spherical cap codes have previously appeared in the literature. They connect the maximum size of codes in a spherical cap, and more generally, in a spherical strip and on the entire sphere.

**3-A. SPHERICAL CAP CODES AND SPHERICAL CODES.** Let  $m(n, d)$  be the maximum number of points in a unit ball in  $\mathbb{R}^n$  that lie at Euclidean distance  $d$  or more apart. Bounds on  $A(n, \theta, \phi)$  are given in the following theorems.

**Theorem 1.** (Sidelnikov [24], Levenshtein [11])

$$A(n, \theta, \phi) \leq m\left(n, \frac{2 \sin(\theta/2)}{\sin \phi}\right).$$

The proof is based on a mapping  $\delta : \text{Cap}(n, \phi) \rightarrow \mathbb{R}^n$  that transforms the cap to the unit ball in  $\mathbb{R}^n$  according to the following rule:

$$\delta(x) = \frac{1}{\sin \phi}(x - e_n \cos \phi).$$

**Theorem 2.** (Levenshtein [12]).

$$m\left(n-1, \frac{2 \sin \theta/2}{\sin \phi \cos \phi}\right) \leq A(n, \theta, \phi) \leq m(n-1, 2 \sin \theta/2 \cot \phi).$$

The proof is based on a mapping that projects the cap centrally on the tangent hyperplane to the sphere  $S^{n-1}$  at the point  $e_n$ .

Bounds of these two theorems are useful in asymptotics (both as  $n \rightarrow \infty$  and as  $\theta \rightarrow 0$ ) for estimating the size of spherical codes [10] and the packing density in  $\mathbb{R}^n$  [24, 12]. Their use for finite  $n$  is based on the obvious inequality  $m(n-1, 2d) \leq A(n, 2 \arcsin d)$  and leads to the estimates

- (1)  $A(n, \theta, \phi) \leq A(n+1, 2 \arcsin(\sin \theta/2 \operatorname{cosec} \phi)) = A(n+1, \omega(\theta, \phi))$
- (2)  $A(n, \theta, \phi) \leq A(n, 2 \arcsin(\sin \theta/2 \cot \phi)).$

For large  $n$  upper bound (1) is uniformly better than bound (2) because  $1/\sin \phi > \cot \phi$ . For finite  $n$  bound (2) is stronger than (1) for small  $\phi$  and is weaker than it otherwise.

**3-B. SPHERICAL STRIP CODES.** In this subsection we discuss the action of the projection  $\Pi_n$  on the code in a spherical strip  $Z(n, [\psi, \phi])$ . Given a  $\theta$ -code  $C \subset Z$  we would like to know what happens to its distance upon applying the mapping  $\Pi_n$  to it. Given two points  $x_1, x_2 \in Z$  the main issue is to establish how the distance between their images depends on their relative location in the strip. Let  $\text{dist}(x_1, x_2) = \theta$  and let the angle  $\gamma \neq \phi, 0 \leq \gamma \leq 90^\circ$  be defined by the equation

$$\cos \omega(\theta, \phi) = \cos \omega(\theta, \phi, \gamma).$$

Geometrically, the angle  $\gamma$  is defined as follows. Consider two points  $x_1, x_2$  that are  $\theta$  away from each other and lie on the boundary of the cap  $\text{Cap}(n, \phi)$  (i.e., the angle between each of them and  $e_n$  is  $\phi$ ). The distance between their images under  $\Pi_n$  equals  $\omega(\theta, \phi)$ . Consider the point  $x'_2$  that satisfies  $\Pi_n(x_2) = \Pi_n(x'_2)$  and  $\langle x_2, x'_2 \rangle = \cos \theta$ , then  $\gamma = \text{dist}(x_2, x'_2)$  (see Figure 1). Formally,  $\gamma$  is the angle given by

$$\sin \gamma = \frac{\sin \phi (\cos^2 \theta - \cos^2 \phi)}{\cos^2 \theta + \cos^2 \phi (1 - 2 \cos \theta)}.$$

Under the mapping  $\Pi_n$  the code in a strip is transformed to a spherical code in  $S^{n-2}$ . This transformation yields nontrivial results only in the case of  $\theta > \psi - \phi$  (otherwise the code  $C$  can contain points that project identically on the equator, so the distance of the image code is zero). In this case, the distance in the image code is minimized for a pair of points on the “lower” boundary of the strip if  $\gamma < \psi$  and for a pair of points one of which is on the lower and the other on the upper boundary, otherwise.

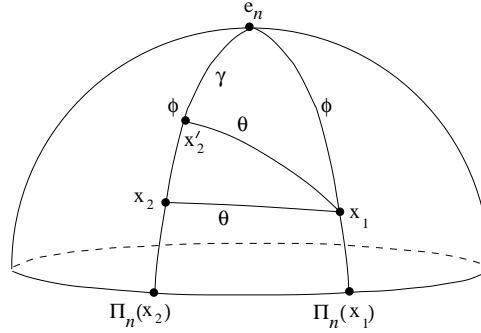


FIGURE 1. Definition of the angle  $\gamma$ : the triangle  $x_1x_2x'_2$  is isosceles.

More precisely, we have the following theorem.

**Theorem 3.** (Agrell, Vardy and Zeger [1]) *Let  $0 \leq \psi \leq \phi \leq \pi/2$  and  $\theta < 2\phi$ .*

(a) *Let  $\theta > \phi - \psi$ . If  $\gamma < \psi$  then*

$$A(n, \theta, [\psi, \phi]) = A(n-1, \omega(\theta, \phi)).$$

*If  $\gamma > \psi$  then*

$$(3) \quad A(n, \theta, [\psi, \phi]) \leq \min \left\{ A(n, \theta, [\psi, \gamma]) + A(n-1, \omega(\theta, \phi)), A(n-1, \omega(\theta, \phi, \psi)) \right\}$$

(b) *Let  $\theta \leq \phi - \psi$ . Then*

$$A(n, \theta, [\psi, \phi]) \leq A(n, \theta, [\psi, \gamma]) + A(n-1, \omega(\theta, \phi)) \quad (\gamma > \psi)$$

*If  $\theta > 2\phi$  then  $A(n, \theta, [\psi, \phi]) = 1$ .*

By taking  $\psi = 0$  this theorem implies the following corollary for spherical cap codes.

**Corollary 1.** [1]

$$(4) \quad A(n, \theta, \phi) \begin{cases} \leq A(n, \theta, \gamma) + A(n-1, \omega(\theta, \phi)) & \text{if } 0 < \theta \leq \phi \\ = A(n-1, \omega(\theta, \phi)) & \text{if } \phi < \theta \leq 2\phi \\ = 1 & \text{if } \theta > 2\phi. \end{cases}$$

3-C. LOWER BOUNDS. A general lower (existence) bound on the size of a  $\theta$ -code in the cap  $\text{Cap}(n, \phi)$  can be obtained by the standard greedy argument. It follows that there exist codes of size

$$M \geq \Omega_n(\phi)/\Omega_n(\theta),$$

where by  $\Omega_n(\beta) = \frac{2\pi^{(n-1)/2}}{\Gamma((n-1)/2)} \int_0^\beta \sin^{n-2} \tau d\tau$  we denote the area of the spherical cap  $\text{Cap}(n, \beta)$  on the sphere  $S^{n-1}$ .

#### 4. AN EXTENSION OF DELSARTE'S METHOD

In this section we explain a way to use bounds on spherical cap codes in order to extend the well-known Delsarte method for bounding the size of spherical codes.

The original Delsarte (linear programming) bound as applied to spherical codes [8, 10] has the following form. Let  $\{G_k^{(n)}(x)\}_{k=0,1,\dots}$  denote the family of Gegenbauer polynomials, i.e., polynomials orthogonal on  $[-1, 1]$  with weight  $(1-x)^{(n-3)/2}$  and

satisfying the normalization condition  $G_k^{(n)}(1) = 1$ . Suppose that a real function  $f$  is a nonnegative linear combination of Gegenbauer polynomials  $G_k^{(n)}(t)$ , i.e.,

$$f(t) = \sum_k f_k G_k^{(n)}(t), \quad \text{where } f_k \geq 0.$$

If  $f(t) \leq 0$  for all  $t \in [-1, \cos \theta]$  and  $f_0 > 0$ , then  $A(n, \theta) \leq f(1)/f_0$ .

Next we consider an extension of this method to spherical caps. Let  $Y = \{y_1, \dots, y_m\}$  be a  $\theta$ -code in the spherical cap  $\text{Cap}(n, \phi)$  with center  $e_n$  and let  $\mathcal{Y}$  be the set of all such codes. Of course,  $m \leq A(n, \theta, \phi)$ . Let  $e_n^* = -e_n$ , let  $f(t)$  be a real function on the interval  $[-1, 1]$ ,

$$H_f(Y) = H_f(y_1, \dots, y_m) := f(1) + f(\langle e_n^*, y_1 \rangle) + \dots + f(\langle e_n^*, y_m \rangle),$$

$$h_m(n, \theta, \phi, f) := \max_{Y \in \mathcal{Y}} \{H_f(Y)\}, \quad h_{\max}(n, \theta, \phi, f) := \max_{m \leq A(n, \theta, \phi)} \{h_m(n, \theta, \phi, f)\}.$$

**Theorem 4.** Suppose that  $f$  is a nonnegative linear combination of Gegenbauer polynomials  $G_k^{(n)}(t)$ , i.e.,

$$f(t) = \sum_k f_k G_k^{(n)}(t), \quad \text{where } f_k \geq 0.$$

If  $f(t) \leq 0$  for all  $t \in [-\cos \phi, \cos \theta]$  and  $f_0 > 0$ , then

$$A(n, \theta) \leq \frac{h_{\max}(n, \theta, \phi, f)}{f_0}.$$

*Proof.* Let  $C = \{x_1, \dots, x_M\}$  be a  $\theta$ -code in  $S^{n-1}$ . It is well known [8, 10] that

$$\sum_{i=1}^M \sum_{j=1}^M G_k^{(n)}(t_{i,j}) \geq 0, \quad t_{i,j} := \langle x_i, x_j \rangle = \cos(\text{dist}(x_i, x_j)).$$

Using this we obtain

$$S_f(C) := \sum_{i=1}^M \sum_{j=1}^M f(t_{i,j}) = \sum_{k,i,j} f_k G_k^{(n)}(t_{i,j}) \geq \sum_{i,j} f_0 G_0^{(n)}(t_{i,j}) = f_0 M^2.$$

Let

$$J(i) := \{j : f(\langle x_i, x_j \rangle) > 0, j \neq i\}, \quad C(i) = \{x_j \in C : j \in J(i)\}, m_i = |C(i)|.$$

Note that  $j \in J(i)$  only if  $x_j$  belongs to the  $\text{Cap}(n, \phi)$  with the center at  $-x_i$ . Then

$$S_i(C) := \sum_{j=1}^M f(\langle x_i, x_j \rangle) \leq f(1) + \sum_{j \in J(i)} f(\langle x_i, x_j \rangle) = H_f(C(i)) \leq h_{m_i}(n, \theta, \phi, f).$$

Therefore,

$$f_0 M^2 \leq S_f(C) = \sum_{i=1}^M S_i(C) \leq M h_{\max},$$

i.e.  $M \leq h_{\max}/f_0$  as required.  $\square$

Note that  $h_0 = f(1)$ . If  $f(t) \leq 0$  for all  $t \in [-1, \cos \theta]$ , then all  $m_i = 0$ , i.e.  $h_{\max} = h_0 = f(1)$  and  $M \leq f(1)/f_0$ , so this theorem includes the Delsarte bound as a particular case.

For given  $n, \theta, \phi, f$  and  $m$  the value  $h_m(n, \theta, \phi, f)$  is the solution of the following optimization problem on  $S^{n-1}$ :

$$h_m(n, \theta, \phi, f) = f(1) + f(-\cos \phi_1) + \dots + f(-\cos \phi_m) \rightarrow \max$$

subject to the constraints

$$\phi_i := \text{dist}(e_n^*, y_i) \leq \phi, \quad 1 \leq i \leq m; \quad \text{dist}(y_i, y_j) \geq \theta, \quad i \neq j.$$

The dimension of this problem is  $m(n-1) \leq (n-1)A(n, \theta, \phi)$ . For relatively small  $n$  and  $A(n, \theta, \phi)$  optimization can be carried out numerically. Moreover, if in addition to the above restrictions the function  $f(t)$  is monotone decreasing for  $t \in [-1, -\cos \phi]$  then in some cases the dimension of this problem can be reduced to  $n$  (see the details in [17, 14]). Suitable polynomials  $f$  can be found by linear programming (see an algorithm in the Appendix to [14]).

## 5. THE CASE OF LARGE ANGLES

In this section we consider  $\theta$ -codes in a spherical cap  $\text{Cap}(n, \phi)$  with large values of  $\theta$ . More precisely let us assume that  $\theta > \phi$ . Clearly, if  $\theta > 2\phi$ , then no more than one point can lie in  $\text{Cap}(n, \phi)$ , i.e.  $A(n, \theta, \phi) = 1$ . Now we consider the case  $2\phi \geq \theta > \phi$ . Recall that  $\phi \leq \pi/2$ .

**Lemma 1.** *Suppose  $2\phi \geq \theta > \phi > 0$ , then  $\omega(\theta, \phi) > \pi/3$ .*

*Proof.* Let  $z := \cos \theta$ ,  $t := \cos \phi$ . Then

$$\cos \omega(\theta, \phi) = \frac{\cos \theta - \cos^2 \phi}{\sin^2 \phi} = \frac{z - t^2}{1 - t^2} \leq \frac{z - z^2}{1 - z^2} = \frac{z}{1 + z} < \frac{1}{2}.$$

Thus,  $\omega(\theta, \phi) > \pi/3$ . □

As stated in Theorem 3, for  $2\phi \geq \theta > \phi$  the problem of finding  $A(n, \theta, \phi)$  is equivalent to bounding the size of spherical codes. Since the proof in [1] is not isolated into a separate argument we include it here for completeness.

**Theorem 5.** (Agrell et al. [1], Musin [14]) *If  $2\phi \geq \theta > \phi$ , then*

$$A(n, \theta, \phi) = A(n-1, \omega(\theta, \phi)).$$

*Proof.* First let us prove the lower bound.

**Lemma 2.** *Let  $0 \leq \psi < \phi \leq \pi/2$ ,  $\theta \leq 2\phi$ . Then*

$$A(n, \theta, \phi) \geq A(n, \theta, [\psi, \phi]) \geq A(n-1, \omega(\theta, \phi)).$$

*Proof.* The first inequality is obvious. To prove the second one let consider the strip  $Z(n, [\psi, \phi]) \subset S^{n-1}$  and let  $\Sigma$  be its “lower” boundary. The projection  $\Pi_n$  is a one-to-one map from  $\Sigma$  to the unit sphere  $S^{n-2}$  (the equatorial sphere of  $S^{n-1}$ ). Now consider a code  $C' \subset S^{n-2}$  and the code  $C \subset \Sigma$  that corresponds to  $C'$  under this map. If the distance of  $C'$  equals  $\omega(\theta, \phi)$ , then the distance of  $C$  is  $\theta$  (the function  $\omega(\theta, \phi)$  is monotone). Since  $|C'| = |C| \leq A(n, \theta, [\psi, \phi])$ , this proves the needed inequality. □

Now let  $C = \{x_1, \dots, x_m\}$  be a  $\theta$ -code in  $\text{Cap}(n, \phi)$ . Then

$$\theta_{i,j} := \text{dist}(x_i, x_j) \geq \theta \quad \text{for } i \neq j.$$

Denote by  $\phi_i$  the angular distance between  $e_n$  and  $x_i$ , where  $e_n$  is the center of  $\text{Cap}(n, \phi)$ . Note that  $\phi_i \leq \phi$ .

Let  $X = \Pi_n(C)$  be the image of  $C$  under the projection on the equator  $S^{n-2}$  of the sphere from its North pole  $e_n$ . Denote by  $\gamma_{i,j} = \text{dist}(\Pi_n(x_i), \Pi_n(x_j))$  be the distance between the images of  $x_i$  and  $x_j$  under the projection. Recall the law of cosines for a spherical triangle. Suppose the two sides are  $a, b$  and the angle between them is  $\psi$ , then the third side  $c$  satisfies

$$(5) \quad \cos c = \cos a \cos b + \cos \psi \sin a \sin b.$$

From this and the inequality  $\cos \theta_{i,j} \leq \cos \theta$ , we get

$$\cos \gamma_{i,j} = \frac{\cos \theta_{i,j} - \cos \phi_i \cos \phi_j}{\sin \phi_i \sin \phi_j} \leq \frac{\cos \theta - \cos \phi_i \cos \phi_j}{\sin \phi_i \sin \phi_j}$$

$$\text{Let } Q(\alpha, \beta) = \frac{\cos \theta - \cos \alpha \cos \beta}{\sin \alpha \sin \beta}, \quad \text{then } \frac{\partial Q(\alpha, \beta)}{\partial \alpha} = \frac{\cos \beta - \cos \theta \cos \alpha}{\sin^2 \alpha \sin \beta}.$$

From this it follows that if  $0 < \alpha, \beta \leq \phi$  then  $\cos \beta \geq \cos \theta$  (because  $\theta \geq \phi$ ); therefore  $\partial Q(\alpha, \beta)/\partial \alpha \geq 0$ , i.e.,  $Q(\alpha, \beta)$  is a monotone increasing function in  $\alpha$ . We have  $Q(\alpha, \beta) \leq Q(\phi, \beta) = Q(\beta, \phi) \leq Q(\phi, \phi)$ . Therefore,

$$\cos \gamma_{i,j} \leq \frac{\cos \theta - \cos \phi_i \cos \phi_j}{\sin \phi_i \sin \phi_j} \leq \frac{\cos \theta - \cos^2 \phi}{\sin^2 \phi} = \cos \omega(\theta, \phi).$$

Thus  $X$  is an  $\omega(\theta, \phi)$ -code on the  $(n-2)$ -sphere. That yields

$$A(n, \theta, \phi) \leq A(n-1, \omega(\theta, \phi)).$$

□

It is proved in [15, 17] that in the case covered by this theorem, points in an extremal configuration are vertices of a convex polyhedron, and lie on the boundary of the cap. This implies that if  $\theta = \phi$  then the code can be augmented by the point  $e_n$  without reducing its distance, so  $A(n, \theta, \theta) = A(n, \omega(\theta, \theta)) + 1$ .

Denote by  $\varphi_n(M)$  the largest angular distance in a spherical code on  $S^{n-1}$  that contains  $M$  points. Recall that  $k(n)$  denotes the kissing number in  $n$  dimensions.

**Corollary 2.** *Suppose that  $\theta > \phi$ , then*

$$A(n, \theta, \phi) \leq k(n-1).$$

Moreover, if  $\varphi_{n-1}(K) \leq \pi/3$ , then

$$A(n, \theta, \phi) < K.$$

*Proof.* By Lemma 1 we can write  $\omega(\theta, \phi) = \pi/3 + \varepsilon$ ,  $\varepsilon > 0$ . Then the theorem yields

$$A(n, \theta, \phi) = A(n-1, \omega(\theta, \phi)) = A(n-1, \pi/3 + \varepsilon) \leq A(n-1, \pi/3) = k(n-1).$$

If  $\varphi_{n-1}(K) \leq \pi/3$ , then  $A(n-1, \pi/3 + \varepsilon) < K$ . □

Using Theorem 5 together with this corollary we can find the exact value  $A(3, \theta, \phi)$  and  $A(4, \theta, \phi)$  for  $\theta > \phi$ . We can also find  $A(n, \theta, \phi)$  for all  $n$  if  $\cos \theta < \cos^2 \phi$ .

1. Let  $n = 3$ . Note that  $k(2) = 6$ ,  $\varphi_2(6) = \pi/3$ , and  $\varphi_2(M) = 2\pi/M$ . Then

$$A(3, \theta, \phi) = \lfloor 2\pi/\omega(\theta, \phi) \rfloor \leq 5.$$



2. Let  $n = 4$ . In three dimensions the best codes and the values  $\varphi_3(M)$  are presently known for  $M \leq 12$  and  $M = 24$  (see [6, 9, 22]). It follows from Fejes Tóth's bound [9] that

$$\varphi_3(2) = 180^\circ, \quad \varphi_3(3) = 120^\circ, \quad \varphi_3(4) = \arccos(-1/3) \approx 109.47^\circ,$$

$$\varphi_3(6) = 90^\circ, \quad \varphi_3(12) = \arccos(1/\sqrt{5}) \approx 63.435^\circ.$$

Schütte and van der Waerden [22] proved that

$$\varphi_3(5) = \varphi_3(6) = 90^\circ, \quad \varphi_3(7) \approx 77.87^\circ \quad (\cos \varphi_3(7) = \cot 40^\circ \cot 80^\circ),$$

$$\varphi_3(8) = \arccos \frac{\sqrt{8}-1}{7} \approx 74.86^\circ, \quad \varphi_3(9) = \arccos \frac{1}{3} \approx 70.53^\circ.$$

The cases  $M = 10, 11$  were considered by Danzer [6]:

$$\varphi_3(10) \approx 66.15^\circ, \quad \varphi_3(11) = \varphi_3(12).$$

Since  $k(3) = 12$  [23], we have  $A(4, \theta, \phi) \leq 12$ . Thus

$$A(4, \theta, \phi) = \max_{M \leq 12} \{M : \varphi_3(M) \geq \omega(\theta, \phi)\}.$$

3. Let  $\cos \theta < \cos^2 \phi$ . In this case we have  $\omega(\theta, \phi) > 90^\circ$ . It is well known [21] that for all dimensions

$$\varphi_n(M) = \arccos \left( \frac{-1}{M-1} \right), \quad 2 \leq M \leq n+1;$$

and

$$\varphi_n(n+2) = \dots = \varphi_n(2n-1) = \varphi_n(2n) = 90^\circ.$$

Therefore, for  $\arccos(\cos^2 \phi) < \theta \leq 2\phi$ , we obtain:

$$A(n, \theta, \phi) = \max_{M \leq n} \{M : \arccos \frac{-1}{M-1} \geq \omega(\theta, \phi)\}.$$

The results obtained can be applied for the kissing number problem as follows. For  $n = 3$  let us consider the following polynomial  $f$ :

$$f(t) = \frac{2431}{80}t^9 - \frac{1287}{20}t^7 + \frac{18333}{400}t^5 + \frac{343}{40}t^4 - \frac{83}{10}t^3 - \frac{213}{100}t^2 + \frac{t}{10} - \frac{1}{200}.$$

This polynomial satisfies the assumptions of Theorem 4 with  $\theta = 60^\circ$ ,  $\phi \approx 53.794^\circ$  ( $f(-\cos \phi) = 0$ ), and  $f_0 = 1$ . In this case  $A(3, \theta, \phi) = 4$ . Since  $h_{\max} < 13$  (see a proof in [18]) we have  $k(3) = A(3, \pi/3) < 13$ , i.e.  $k(3) = 12$ .

In the case of  $n = 4$ , Theorem 4 can be applied with

$$f(t) = 53.76t^9 - 107.52t^7 + 70.56t^5 + 16.384t^4 - 9.832t^3 - 4.128t^2 - 0.434t - 0.016.$$

Here  $\theta = 60^\circ$ ,  $\phi \approx 52.559^\circ$ ,  $f_0 = 1$ , and  $A(4, \theta, \phi) = 6$ . It was proved [14] that  $h_{\max} < 25$ . Since  $k(4) \geq 24$  this yields  $k(4) = 24$ .

Recently, Pfender [20] considered the case  $\cos \theta < \cos^2 \phi$ . He found some improvements for upper bounds on  $k(n)$  for dimensions  $n = 9, 10, 16, 17, 25, 26$ .

## 6. STRETCHING TRANSFORMATION

In this section we will prove the following bound on spherical cap codes which relates the maximum size of such a code to the size of codes in a hemisphere.

**Theorem 6.** *Let  $\theta/2 < \phi \leq \pi/2$ . Then*

$$(6) \quad A(n, \theta, \phi) \leq B(n, \omega(\theta, \phi)).$$

*Remark.* This theorem improves upon the bound (1) by reducing the dimension on the right-hand side by one. It also extends the applicability of the bound in Theorem 5 to the range of angles  $\theta/2 \leq \phi \leq \theta$ , although in this range we cannot claim the exact equality anymore. On the other hand, by Theorem 6 it is sufficient to estimate the number of code points in the hemisphere as opposed to the entire sphere.

Note also that for any  $\theta, \phi$ ,

$$\cos \omega(\theta, \phi) - \cos \theta = \tan^2 \phi (\cos \theta - 1) < 0$$

therefore, this theorem is stronger in the entire range of angles than the trivial bound  $A(n, \theta, \phi) \leq A(n, \theta)$ . Finally, the angle  $\omega(\theta, \phi)$  ranges between  $\pi$  and  $\theta$  as  $\phi$  grows from  $\theta/2$  to  $\pi/2$  and is a monotone decreasing function of  $\theta$ .

A proof of Theorem 6 will follow from the following result which describes the effect on the distance of spherical cap codes of a “stretching map” of spherical caps.

**Theorem 7.** *Let  $0 < \phi \leq \pi/2$ . Then for any  $s \geq 1$*

$$A(n, \theta, \phi) \leq A(n, \theta', s\phi),$$

where

$$\cos \theta' = \cos^2 s\phi + \frac{\sin^2 s\phi}{\sin^2 \phi} (\cos \theta - \cos^2 \phi).$$

*Proof:* Let  $T_s, s \geq 1$  be a map on  $\text{Cap}(n, \phi)$  defined as follows: for a point  $x \in \text{Cap}(n, \phi)$  with  $\text{dist}(x, e_n) = \alpha$  its image  $\mathbf{y} = T_s(x)$  is a point that satisfies  $\text{dist}(\mathbf{y}, e_n) = s\alpha$  and lies on the meridian passing through  $x$ . Thus,  $T_s(\text{Cap}(n, \phi)) = \text{Cap}(n, s\phi)$ , and we assume that  $s\phi \leq \pi/2$ .

The proof of Theorem 7 relies upon the next two lemmas.

**Lemma 3.** *Let  $x_1, x_2 \in \text{Cap}(n, \phi)$  with  $\text{dist}(x_1, e_n) = u, \text{dist}(x_2, e_n) = v, \text{dist}(x_1, x_2) = \theta$ . The distance  $\text{dist}(T_s(x_1), T_s(x_2))$  reaches its minimum when  $u = v$ .*

*Proof.* Figure 2 shows the relative location on the sphere of  $x_1, x_2$  and their images  $\mathbf{y}_1 = T_s(x_1), \mathbf{y}_2 = T_s(x_2)$ .

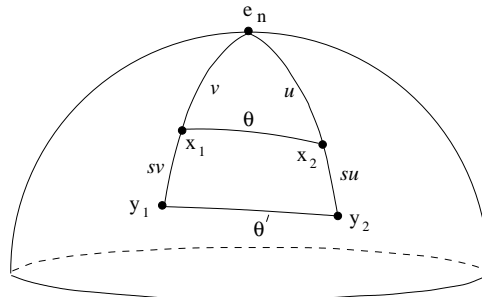


FIGURE 2.

Using (5) we find that  $\cos \theta' = F(u, v, \theta)$ , where

$$F(u, v, \theta) = \cos su \cos sv + \rho(u)\rho(v)(\cos \theta - \cos u \cos v),$$

$\rho(t) = \sin(st)/\sin t$ . For definiteness assume that  $v \leq u$ . We need to prove that  $F(u, v, \theta) \leq F(u, u, \theta)$  where  $0 \leq u - v \leq \theta$  and  $su \leq \pi/2$ .

*Fact 1.* (i) The function  $\rho(t)$  is monotone decreasing for  $t \in (0, \pi/2s)$ . Indeed

$$\rho'(t) = \frac{\cos t \cos st}{\sin^2 t}(s \tan t - \tan st) \leq 0$$

with the equality only if  $t = 0$ .

(ii)  $\rho(t) < s$  (follows from (i) and the equality  $\rho(0) = s$ ).

*Fact 2.* The function  $S(\theta) = F(u, v, \theta) - F(u, u, \theta)$  is maximized on  $\theta$  for  $\theta = u - v$ .

*Proof:* The coefficient of  $\cos \theta$  in  $S(\theta)$  equals  $\rho(u)(\rho(v) - \rho(u)) \geq 0$ . Then the claim follows from the condition  $\pi/2 \geq \theta \geq u - v$ .

*Fact 3* (which implies the lemma).  $S(\theta) \leq S(u - v) \leq 0$ .

*Proof:* The first inequality is proved in Fact 2. Now compute

$$\begin{aligned} S(u - v) &= \cos s(u - v) - \cos^2 su - \frac{\sin^2 su}{\sin^2 u}(\cos(u - v) - \cos^2 u) \\ &= \cos s(u - v) - 1 - \rho^2(u) \cos(u - v) + \rho^2(u). \end{aligned}$$

The derivative of the last expression on  $v$  equals  $(s\rho(u - v) - \rho^2(u)) \sin(u - v)$ . Since  $\rho(u) \leq \rho(u - v)$ , we can write

$$(s\rho(u - v) - \rho^2(u)) \sin(u - v) \geq (s - \rho(u))\rho(u - v) \sin(u - v) \geq 0$$

where the last inequality follows from part (ii) of Fact 1 above.  $\square$

**Lemma 4.** Let  $x_1, x_2 \in \text{Cap}(n, \phi)$  with  $\text{dist}(x_1, e_n) = \text{dist}(x_2, e_n) = u$ ,  $\text{dist}(x_1, x_2) = \theta$ , where  $u \leq \phi$ . Then the distance  $\text{dist}(T_s(x_1), T_s(x_2))$  reaches its minimum when  $u = \phi$ .

*Proof.* Since

$$\cos \theta' = F(u, u, \theta) = 1 - \rho^2(u)(1 - \cos \theta),$$

the claim is implied by Fact 1(i) above.  $\square$

The last two lemmas imply Theorem 7. Indeed, let  $x_1, x_2 \in \text{Cap}(n, \theta, \phi)$  be two points at distance  $\theta$ . The lemmas show that in order for the distance between their images under  $T_s$  to reach its minimum the points should lie on the boundary of the cap. Then the expression for  $\theta'$  in the theorem is implied by an application of the cosine law (5).

Finally, Theorem 6 follows from Theorem 7 by taking  $s = \pi/2\phi$ .  $\square$

We conclude this section with two applications of Theorem 6.

6-A. AN UPPER BOUND ON SPHERICAL CODES. Here we establish the following new estimates:

$$(7) \quad A(n, \theta) \leq \frac{\Omega_n}{\Omega_n(\phi)} B(n, \omega(\theta, \phi)).$$

$$(8) \quad A(n, \theta) \leq \frac{\Omega_n}{\Omega_n(\phi)} A(n - 1, \omega(\theta, \phi)) \quad (\theta > \phi)$$

where  $\Omega_n(\phi)$  is the area of the cap of radius  $\phi$  and  $\Omega_n = \pi^{n/2}/\Gamma(n/2 + 1)$  is the “surface area” of the unit sphere  $S^{n-1}$ . They are implied by the Bassalygo-Elias inequality stated in the next lemma.

**Lemma 5.** [24, 11] *Let  $\theta/2 \leq \phi \leq \pi/2$ . Then*

$$A(n, \theta) \leq \frac{\Omega_n}{\Omega_n(\phi)} A(n, \theta, \phi).$$

*Proof:* Consider a code  $C \subset S^{n-1}$  and let  $C_\phi(z)$  be the number of code points in the cap with “center”  $z$  and radius  $\phi$ . Note that every cap whose center  $z$  is at most  $\theta$  away from a given code point  $x$  will contain this point. Then clearly

$$\int_{z \in S^{n-1}} |C_\phi(z)| dz = \Omega_n(\phi) |C|.$$

Since  $C_\phi(z) \leq A(n, \theta, \phi)$ , we obtain

$$A(n, \theta, \phi) \Omega_n \geq A(n, \theta) \Omega_n(\phi).$$

□

Therefore, using Theorems 6 and 5 we obtain the bounds (7), (8). In particular, inequality (7) is stronger than bounds on cap codes based on Lemma 5 that appeared in [24, 11, 12].

6-B. LARGE DIMENSIONS. Let  $R(C) = 1/n \ln |C|$  be the *rate* of the code  $C \subset S^{n-1}$ . Denote by

$$R^+(\theta) = \limsup_{n \rightarrow \infty} \frac{1}{n} \ln A(n, \theta) \quad R^-(\theta) = \liminf_{n \rightarrow \infty} \frac{1}{n} \ln A(n, \theta).$$

Abusing notation, we write below  $R(\theta)$  to refer to the common value of  $R^+$  and  $R^-$  even though it is not known that the limit exists. Likewise we write  $R(\theta, \phi)$  and  $R(\theta, [\psi, \phi])$  to refer to cap and strip codes. In this section we show that the problem of finding either of the last two quantities is equivalent to that of computing  $R(\theta)$ . More precisely, we have the following theorem.

**Theorem 8.** *Let  $0 \leq \psi < \phi \leq \pi/2$ . Then*

$$R(\theta, [\psi, \phi]) = R(\theta, \phi) = R(\omega(\theta, \phi)).$$

Indeed, combining Lemma 2, Theorem 6 and the obvious inequality  $B(n, \alpha) \leq A(n, \alpha)$  we obtain the bounds

$$(9) \quad A(n-1, \omega(\theta, \phi)) \leq A(n, \theta, [\psi, \phi]) \leq A(n, \theta, \phi) \leq A(n, \omega(\theta, \phi))$$

which imply this theorem.

Inequality (9) can be rewritten in a somewhat more visual way. Consider two points  $x_1, x_2 \in \text{Cap}(n, \phi)$  with angular distance  $\theta$ . Let  $d$  be the Euclidean distance between them. From (5) the angular distance between their images under  $\Pi_n$  is  $\theta' \leq \omega(\theta, \phi)$  and the Euclidean distance equals  $2 \sin \theta'/2 \leq d/\sin \phi$ . Hence the minimum distance of the image code  $\Pi_n(C) \subset S^{n-2}$  is at most  $d/\sin \phi$ . Denote by  $N(n, d, \phi) = A(n, \arccos(1 - d^2/2), \phi)$  the maximum number of points in a spherical cap code with minimum Euclidean distance  $d$  and let  $N(n, d)$  be the same for the sphere. An equivalent form of (9) for spherical caps is as follows:

$$N(n-1, d/\sin \phi) \leq N(n, d, \phi) \leq N(n, d/\sin \phi).$$

A somewhat weaker upper bound on  $A(n, \theta, \phi)$  than in (9) is given by (1). This bound is nevertheless sufficient to establish one part of the asymptotic claim of Theorem 8.

## 7. CODES IN HEMISPHERES

Here we consider upper bounds on  $B(n, \theta) = A(n, \theta, \pi/2)$ . Let  $C \subset S_+ = \text{Cap}(n, \pi/2)$  be a  $\theta$ -code in the hemisphere. Denote

$$(10) \quad C([\alpha, \beta]) = C \cap Z(n, [\alpha, \beta]), \quad C(\alpha) = C \cap \text{Cap}(n, \alpha).$$

**Theorem 9.** *Let  $\theta < \pi/2$ ,  $\delta = (\pi - \theta)/2$  and let  $C \subset S_+$  be a  $\theta$ -code. Then*

$$|C([\delta, \pi/2])| + 2|C(\delta)| \leq A(n, \theta).$$

*Proof.* Let  $a = |C([\delta, \pi/2])|$ ,  $b = |C(\delta)|$ . For a point  $x = (x_1, \dots, x_{n-1}, x_n)$  denote by  $x^* = (x_1, \dots, x_{n-1}, -x_n)$  its reflection about the equator. Let  $C^*(\delta) = \{x^* : x \in C(\delta)\}$  be the reflection of the code  $C(\delta)$ . Consider the code  $Q = C \cup C^*(\delta)$ . We claim that  $Q$  is a  $\theta$ -code. Referring to Fig. 3 this amounts to showing that  $\text{dist}(q, p^*) \geq \theta$  if  $\text{dist}(q, p) \geq \theta$ . To prove this, we choose the point  $s$  so that the angle  $\angle qsp = 90^\circ$  and use (5) as follows:

$$\cos \theta \geq \cos \alpha = \cos \eta \cos \beta \geq \cos \eta \cos \beta^* = \cos \alpha^*.$$

This proves that  $Q$  is a  $\theta$ -code. Then  $a + 2b = |Q| \leq A(n, \theta)$ .  $\square$

**Corollary 3.**

$$B(n, \theta) \leq 1/2(A(n, \theta, [(\pi - \theta)/2, \pi/2]) + A(n, \theta))$$

*Proof.* Using the notation of the previous theorem, we have

$$2|C| = 2a + 2b \leq a + A(n, \theta) \leq A(n, \theta, [\delta, \pi/2]) + A(n, \theta).$$

$\square$

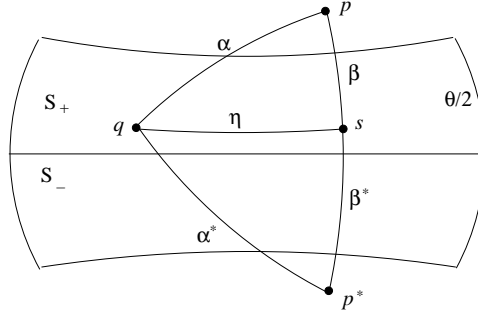


FIGURE 3.

**Corollary 4.**

$$(11) \quad B(n, \theta) \leq \frac{A(n-1, \tilde{\theta}) + A(n, \theta)}{2}, \quad \cos \tilde{\theta} = \frac{\cos \theta}{\cos \theta/2}.$$

*Proof.* We use Theorem 3. In our case  $\phi = \pi/2$ ,  $\psi = (\pi - \theta)/2$ , therefore,  $\theta > \phi - \psi > \theta/2$ , so part (a) of this theorem applies. Substituting the values of  $\phi, \psi$  in the inequality  $A(n, \theta, [\psi, \phi]) \leq A(n-1, \omega(\theta, \phi, \psi))$ , we obtain  $A(n, \theta, [\psi, \pi/2]) \leq A(n-1, \tilde{\theta})$ . The proof is concluded by using this estimate in Corollary 3.  $\square$

Using the value  $\theta = 60^\circ$ , we obtain estimates on the one-sided kissing number  $B(n)$ . In particular, since  $A(n, 60^\circ, 30^\circ) = 2$ , the last corollary yields

**Corollary 5.**

$$B(n) \leq \min[(1/2)(A(n-1, \eta_0) + k(n)), k(n) - 2], \quad \eta_0 := \arccos \frac{1}{\sqrt{3}} \approx 54.74^\circ.$$

Clearly,  $B(2) = 4$ . Let us use this bound for  $n = 3, 4$ .

$n = 3$ . We have  $k(3) = 12$ ,  $A(2, \eta_0) = 6$ , then  $B(3) \leq 9$ . On the other hand,  $B(3) \geq 9$ , so  $B(3) = 9$ . Note that in this case the bound is sharp.

$n = 4$ . Recently, K. Bezdek [2, 3] proved that  $B(4) = 18$  or 19, and conjectured that  $B(4) = 18$ . It was proved, also recently [16, 14], that  $k(4) = 24$ . Delsarte's linear programming method gives  $A(3, \eta_0) \leq 15$ . Thus  $B(4) \leq 19$ . The proof that  $B(4) = 18$  given in [15] is based on an extension of Delsarte's method.

For higher dimensions we can rely on the known bounds for spherical codes. Denote by  $\hat{g}_n$  an upper bound on  $A(n, \eta_0)$  given by Delsarte's linear programming method, and by  $\hat{k}_n$  the known upper bounds on  $k(n)$  (see, e.g., Table 1.5 in [5]). Then Corollary 5 implies

$$B(n) \leq \frac{\hat{g}_{n-1} + \hat{k}_n}{2}.$$

This gives the following bounds:

$$B(5) \leq 39, \quad B(6) \leq 75, \quad B(7) \leq 135, \quad B(8) \leq 238.$$

For instance, for  $n = 8$  we have  $\hat{g}_7 = 236$  (obtained by Delsarte's method with a polynomial of degree 11) and  $k_8 = 240$ . Note that  $\hat{g}_{n-1} > \hat{k}_n$  for  $n > 8$ ; therefore, for these dimensions we just have the bound  $B(n) \leq k(n) - 2$ .

However, even for  $5 \leq n \leq 8$  these bounds are not sharp. Our conjectures for  $n = 5, 8$  are

$$B(5) = 32, \quad B(8) = 183.$$

## 8. A BOUND ON SPHERICAL CAP CODES

In this section the methods and results developed above will be used to derive another bound on spherical cap codes. Given a code  $C \subset \text{Cap}(n, \phi)$ , our plan is to first map the cap on the hemisphere  $S_+$  and then use the results of the previous section together with some additional ideas. In the next theorem we use notation (10).

**Theorem 10.** *Let  $0 \leq \phi \leq \pi/2$ ,  $\omega_1 := \omega(\theta, \phi)$ ,  $\omega_2 := \omega(\theta, \phi, \psi)$ , where  $\psi = \phi(1 - \omega_1/\pi)$ . Then for any  $\theta$ -code  $C \subset \text{Cap}(n, \phi)$  we have*

$$(12) \quad |C([\psi, \phi])| \leq A(n-1, \min(\omega_1, \omega_2))$$

and

$$(13) \quad |C([\psi, \phi])| + 2|C(\psi)| \leq A(n, \omega_1).$$

*Proof.* Consider the mapping  $T_s, s = \pi/2\phi$  that sends the cap to the hemisphere  $S_+$ . By Theorem 7, the code  $C$  is mapped to a code  $T_s(C) \subset S_+$  with distance  $\omega_1$ . Now (13) is implied by Theorem 9.

To prove (12), let us bound above  $|C([\psi, \phi])|$  in terms of  $\theta$  and  $\phi$ . Consider the action on the code  $C([\psi, \phi])$  of the orthogonal projection  $\Pi_n$  on the sphere from  $e_n$  on the equator. The code  $D = \Pi_n(C([\psi, \phi]))$  is a spherical code in  $n-1$  dimensions. Given two points  $x, y \in C([\psi, \phi])$  such that  $\text{dist}(x, e_n) = \alpha_1, \text{dist}(y, e_n) = \alpha_2, \text{dist}(x, y) = \beta$ , the distance between their images under  $\Pi_n$

is given by  $\omega(\beta, \alpha_1, \alpha_2)$ . As shown in the proof of Lemma 2, this function is monotone on  $\alpha_1$  (or  $\alpha_2$ ) if the other two arguments are fixed, so its minimum is attained at one of the boundaries. Therefore, the distance of the code  $D$  is determined according to one of the following two cases:

- (i) There are two points  $c_1, c_2 \in C$  lying on the boundary of the cap and  $\theta$  away from each other. Their images in the code  $D$  are two points on the equator at distance  $\omega_1$ .
- (ii) There are two points  $c_1, c_2$  in  $C$  at distance  $\theta$  such that  $T_s(c_1) = x_1$ ,  $T_s(c_2) = x_2$  (see Fig. 4).

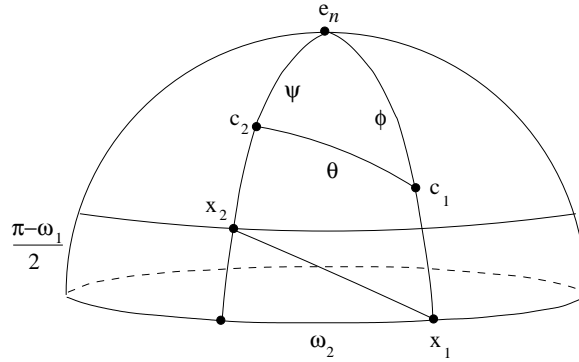


FIGURE 4.

Upon projecting  $x_2$  on the equator, the distance  $\text{dist}(\Pi_n(x_2), x_1) = \omega(\theta, \phi, \psi)$ , where  $\psi = \text{dist}(e_n, c_2)$  satisfies

$$\psi s = \frac{\pi - \omega_1}{2},$$

i.e.,  $\text{dist}(\Pi_n(x_2), x_1) = \omega_2$ .

Therefore,  $D$  is a code with distance at least  $\min(\omega_1, \omega_2)$ .  $\square$

**Corollary 6.** For all  $0 < \theta < \phi < \pi/2$

$$(14) \quad A(n, \theta, \phi) \leq 1/2(A(n-1, \min(\omega_1, \omega_2)) + A(n, \omega_1)).$$

*Proof.* Use (12) and (13) to compute

$$2|C([\psi, \phi])| + 2|C(\phi)| = |C([\psi, \phi])| + |C(\phi)| + |C(\psi)| \leq A(n-1, \min(\omega_1, \omega_2)) + A(n, \omega_1).$$

This gives the claimed result.  $\square$

Comparison of this bound with the other bounds considered in this paper is difficult in general. However, it is clear, that it improves upon the bound (6) in all cases when  $\omega_1 \leq \omega_2$ . The domain of values of  $\theta, \phi$  for which this holds true is shown in Fig. 5.

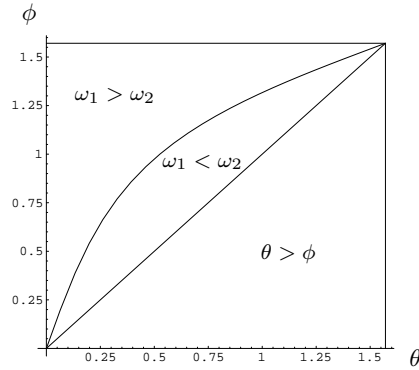


FIGURE 5. Area of improvement for bound (14)

**Examples.** Next we give several examples of the bounds known earlier and obtained in this paper, showing that the results of this paper improve the known estimate in some range of values of  $\theta$  and  $\phi$ .

Bounds on  $A(6, \theta, \phi)$ 

$\theta$	$\phi$	Corollary 1	Theorem 6	Corollary 6
$0.02\pi$	$0.2\pi$	9069268	7581886	4058040
$0.05\pi$	$0.25\pi$	145587	194908	111241
$0.2\pi$	$0.4\pi$	661	832	591
$0.25\pi$	$0.4\pi$	221	272	201
$0.3\pi$	$0.47\pi$	174	138	115

Bounds on  $\log_2 A(n, \theta, \phi)$ 

$n$	$\theta$	$\phi$	Corollary 1	Theorem 6	Corollary 6
20	$0.05\pi$	$0.4\pi$	66.1912	65.7786	65.3665
20	$0.08\pi$	$0.4\pi$	52.7212	52.891	52.6557
30	$0.05\pi$	$0.4\pi$	97.75	97.7008	97.8523
40	$0.05\pi$	$0.45\pi$	132.516	131.447	133.45
100	$0.05\pi$	$0.45\pi$	322.205	321.917	333.28

To compute the bounds in examples one needs to use some upper bound on  $A(n, \theta)$ . We have used the bound of Levenshtein [12], [13, p.618] which is the best known universal upper bound on spherical codes.

## 9. A GENERAL METHOD OF BOUNDING THE SIZE OF CAP CODES

In this section we generalize the method of Sections 7, 8 to develop a general approach to bounding the size of codes in spherical caps.

Let  $C \in \text{Cap}(\theta, \phi)$  be a  $\theta$ -code. Let  $0 = \phi_0 < \phi_1 < \dots < \phi_k < \phi_{k+1} = \phi$ .

To bound above the size of the code  $C$  we partition the cap as follows:

$$\text{Cap}(n, \phi) = \text{Cap}(n, \phi_1) \cup Z(n, [\phi_1, \phi_2]) \cup \dots \cup Z(n, [\phi_k, \phi_{k+1}]).$$



Let  $p_i := |C \cap Z(n, [\phi_i, \phi_{i+1}])|$  be the size of the code in the strip which can be estimated by the methods of [1] and this paper. In particular, denote by  $a_{i,j}$  an upper bound on the size of the code in the strip  $Z(n, [\phi_i, \phi_j])$  and let  $a_i := a_{i,i+1}$ . Suppose that we can compute these estimates for some subset  $P$  of pairs  $(i, j)$ . We have the following linear constraints.

$$(15) \quad 0 \leq p_i \leq a_i, \quad (i, i+1) \in P; \quad p_i + \dots + p_{j-1} \leq a_{i,j}, \quad (i, j) \in P.$$

Another set of linear inequalities can be obtained from the arguments similar to the proof of Theorems 9, 10. Namely, let  $r(i) = \max\{j : s\phi_j \leq (\pi - \omega(\theta, \phi_{i+1}))/2\}$ . Consider the codes  $C' = C(\phi_{r(i)+1})$  and  $C'' = C([\phi_{r(i)+1}, \phi_{i+1}])$ . Upon stretching the cap to the hemisphere by applying the mapping  $T_s$  and using (13), we obtain  $2|C'| + |C''| \leq b_i$ , where  $b_i$  is the size of spherical code with distance  $\omega(\theta, \phi_{r(i)+1})$ . We then obtain inequalities

$$(16) \quad 2(p_0 + p_1 + \dots + p_{r(i)}) + p_{r(i)+1} + \dots + p_i \leq b_i, \quad i \in \Phi,$$

where  $\Phi$  is a subset of indices for which we perform the described procedure.

We summarize the arguments of this section in the following

**Theorem 11.** *The size of the code  $C$  is bounded above by the solution of the linear programming problem*

$$p_0 + p_1 + \dots + p_k \rightarrow \max$$

*under the constraints (15), (16).*

Note that Theorem 10 constitutes a solution of this problem in the case  $k = 0$ .

The set  $\{\phi_i\}$  can be optimized in computations. It can be formed, for instance, by taking all the angles  $\gamma$  produced by the recurrent calculation in Corollary 1 together with some additional breakpoint angles.

## 10. APPLICATIONS

We have discussed the applications of the bounds on codes in caps to the kissing number and the one-sided kissing number problems. As remarked in the introduction, the methods developed in this paper can be also useful in the problems of estimating the packing density in  $\mathbb{R}^n$  and of the size of constant weight codes. We end the paper with brief remarks on these applications.

**10-A. SPHERICAL CODES AND PACKING DENSITY.** Let  $\Delta_n$  be the density of packing the  $n$ -dimensional real space with equal nonoverlapping balls. A classical problem in geometry is to compute  $\Delta_n$  for a given  $n$  and for  $n \rightarrow \infty$ . It is known [5, p.265] that

$$(17) \quad \Delta_n \leq 1/2(\sin \theta/2)^n A(n+1, \theta) \quad (0 < \theta \leq \pi).$$

Using Lemma 5 we now obtain the estimate

$$(18) \quad \Delta_n \leq 1/2 (\sin \theta/2)^n \Omega_{n+1} \frac{A(n+1, \theta, \phi)}{\Omega_{n+1}(\phi)} \quad (0 < \theta \leq \pi, \theta/2 \leq \phi \leq \pi/2).$$

To compute upper bounds on  $\Delta_n$  for a given value of  $n$  we have a choice of using (17) with the known bounds on  $A(n+1, \theta)$ , or using (18) together with the bounds on spherical cap codes considered in this paper. Observe that the best known estimate of the packing density for  $n \rightarrow \infty$  [10] is obtained by employing (18) together with the bound (1) on cap codes. Therefore, inequality (18) coupled with better bounds on cap codes derived in this paper will also improve the density estimates for finite (but possibly large) values of  $n$ .

10-B. CONSTANT WEIGHT CODES. A constant weight binary code is a subset of  $\{0, 1\}^n$  formed of vectors with a fixed number, say  $w$ , of ones. Denote by  $A(n, d, w)$  the maximum size of a constant weight code with minimum Hamming distance  $d$ . Computing or estimating the numbers  $A(n, d, w)$  is a problem with a long history in coding theory, summarized in [1]. The most studied region of parameters is  $n \leq 28$  for which the most recent tables are published in [1]. For larger  $n$  bounds on  $A(n, d, w)$  are tabulated in [25]. In [1] the problem of bounding above  $A(n, d, w)$  was reduced to bounds on spherical cap codes which led to several improvements of the tables for short lengths. We intend to use the new bounds on cap codes derived in this paper to further improve the tables.

As a final remark, note that all the bounds on cap codes considered in this paper rely on bounds on the maximum size  $A(n, \theta)$  of spherical codes. In calculations, apart from the Levenshtein bound and related results it is possible to use a direct solution of Delsarte's linear programming problem relying on a method developed in [19].

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